

# JRF Mathematics Examination

## RM I

Solve any six questions.

1. Suppose that  $X$  is a Hausdorff space and  $\alpha : [0, 1] \rightarrow X$  is a continuous function. If  $\alpha$  is one-one, then prove that the image of  $\alpha$  is homeomorphic to  $[0, 1]$ . Give an example where  $\alpha$  is not one-one but the image of  $\alpha$  is homeomorphic to  $[0, 1]$ .
2. Prove that the following collection of subsets defines a topology on the set of natural numbers  $\mathbb{N}$  :

$$\emptyset, \mathbb{N}, U_n = \{1, \dots, n\}, \quad n \in \mathbb{N}.$$

Is  $\mathbb{N}$  compact in this topology? What are the continuous functions from this space to the space  $\mathbb{R}$  of real numbers with standard topology?

3. Let  $\{a_n\}$  be a sequence of non-zero real numbers. Show that it has a subsequence  $\{a_{n_k}\}$  such that  $\lim \frac{a_{n_{k+1}}}{a_{n_k}}$  exists and belongs to  $\{0, 1, \infty\}$ .
4. Let  $D$  denote the open ball of unit radius about origin in the complex plane  $\mathbb{C}$ . Let  $f$  be a continuous complex-valued function on its closure  $\overline{D}$  which is analytic on  $D$ . If  $f(e^{it}) = 0$  for  $0 < t < \frac{\pi}{2}$ , show that  $f(z) = 0$  for all  $z$ .
5. Let  $U$  be an open connected set in  $\mathbb{C}$  and  $f : U \rightarrow \mathbb{C}$  be a continuous map such that  $z \mapsto f(z)^n$  is analytic for some positive integer  $n$ . Prove that  $f$  is analytic.

6. Let  $C$  be a closed convex set in  $\mathbb{R}^2$ . For any  $x \in C$ , define

$$C_x = \{y \in \mathbb{R}^2 \mid x + ty \in C, \forall t \geq 0\}.$$

Prove that for any two points  $x, x' \in C$ , we have  $C_x = C_{x'}$ .

7. Let  $U = B(c, R)$  denote the open ball of radius  $R$  around  $c \in \mathbb{C}$ . Let  $f$  be an analytic function on  $U$  such that there exists  $M > 0$  with  $|f'(z)| \leq M$  for all  $z \in U$ . Prove that

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2| \quad \forall z_1, z_2 \in U.$$

8. Let  $a, b$  be two real numbers. If the function  $x(t)$  is a solution of

$$\frac{d^2x}{dt^2} - 2a\frac{dx}{dt} + bx = 0$$

with  $x(0) = x(1) = 0$ , then show that  $x(n) = 0$  for all  $n \in \mathbb{N}$ .

(Hint: Show that there exists a constant  $c$  such that  $x(t+1) = cx(t)$  for all  $t$ .)

9. Let  $\mu$  be a measure on the Borel  $\sigma$ -field of  $\mathbb{R}$  such that  $\mu(\mathbb{R}) = 1$ . Recall that the support of  $\mu$  is the largest closed set  $C$  such that for all open sets  $U$  with  $U \cap C \neq \emptyset$ , we have  $\mu(U) > 0$ . Assume that every continuous real-valued function on  $\mathbb{R}$  is integrable with respect to  $\mu$ . Prove that the support of  $\mu$  is compact.
10. Find all pairs  $(x_0, y_0) \in \mathbb{R}^2$  such that the following initial value problem has a unique solution in a neighbourhood of  $(x_0, y_0)$ :

$$y' = y^{\frac{1}{3}} + x; \quad y(x_0) = y_0.$$

11. Let  $(X, d)$  be a complete metric space. Let  $f : X \rightarrow X$  be a function such that for all distinct  $x, y \in X$ ,

$$d(f^k(x), f^k(y)) < c \cdot d(x, y),$$

for some real number  $c < 1$  and an integer  $k > 1$ . Show that  $f$  has a unique fixed point.

(Here  $f^r$  denotes the  $r$ -fold composition of  $f$  with itself.)